Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser, bind **30**, nr. 10

Dan. Mat. Fys. Medd. **30**, no. 10 (1955)

DEDICATED TO PROFESSOR NIELS BOHR ON THE OCCASION OF HIS 70TH BIRTHDAY

OLD PROBLEMS IN THE GENERAL THEORY OF RELATIVITY VIEWED FROM A NEW ANGLE

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København 1955
i kommission hos Ejnar Munksgaard

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Printed in Denmark
Bianco Lunos Bogtrykkeri A-S

Introduction.

It is a well known consequence of basic assumptions in the general theory of relativity that the rate of an ideal standard clock moving with the velocity v through a gravitational field with the potential χ is determined by the formula

$$d\tau = dt\sqrt{1 + 2\chi/c^2 - v^2/c^2}, \tag{1}$$

where τ is the proper time of the standard clock and t is the coordinate time in a time-orthogonal system of space-time coordinates[1]. Equation (1) is equivalent to the statement that the proper time of a particle is a measure of the length of the time track of the particle in $(3 + 1)$-space. It follows directly from the principle of relativity and the equivalence of gravitational fields and "acceleration fields", together with the assumption that the rate of the standard clock is equal to the rate of the clocks in a local rest system of inertia. The last assumption implies that the acceleration of an ideal standard clock relative to a system of inertia has no influence on the rate of the clock, which thus is entirely determined by its velocity.

The formula (1) is closely connected with the well-known formula for the red-shift of spectral lines emitted by atoms situated at places with a negative gravitational potential, and gives the clue also to a solution, of the so-called clock paradox[2]. On account of the inherent invariance of the length of the time track of a particle, it is clear beforehand that no real contradictions connected with the rate of moving clocks can ever arise in this theory.

However, just for this reason, students of the theory of relativity very often do not find the usual solution of the clock paradox satisfactory. They maintain—rightly of course—that one has made just such *assumptions* about the behaviour of clocks in gravitational fields that no paradox can occur, and they would like to see a derivation of (1) on the basis of the dynamical laws governing the functioning of a clock or at least of a simple model of a clock. This desire is of a similar kind as that which, in the early days of relativity theory, led to attempts at deriving the Lorentz contraction of moving rigid bodies from the laws governing the constitution of solid bodies. Against such attempts it has been objected that the effects in question are much more elementary and much more directly connected with the principles of the theory than the laws from which they are proposed to be derived, so that the behaviour of moving rigid bodies and standard clocks rather represents a challenge to the theory of the constitution of matter and to the dynamical laws underlying the functioning of clocks. This is certainly a sound objection in the case of the contraction phenomena, since we do not at the moment have a consistent relativistic atomic theory of solid bodies from which the contraction phenomenon could be deduced. However, the situation is somewhat different in the case of the formula (1) for two reasons. Firstly, a clock may be in a certain sense regarded as a much simpler system than a measuring rod, since, for instance, any macroscopic particle performing harmonic oscillations around a centre under the influence of elastic forces may be used as a clock. Thus, in calculating the rate of such a clock, we can neglect all quantal effects and we need only a knowledge of the dynamical laws governing the motion of a macroscopic particle acted upon by an external gravitational field and by a given non-gravitational force. Secondly, as shown by Einstein, Infeld, and Hoffmann[3)], these dynamical laws follow from the gravitational field equations without further assumptions. In particular, it was shown by Infeld and Schild[4)] that the time track of a freely falling test particle (that is, a particle of vanishing mass) in an arbitrary gravitational background field, is bound to be a geodesic line in the space-time continuum of the background field in order that the field equations can have solutions. This interesting theorem, according to which the equations of motion appear as a kind of integrability conditions for the field equations, is closely connected with the non-linear character of EINSTEIN's field equations. It is true that the correct equations of motion for a freely falling particle had been derived already long ago on the basis of the principles of relativity and equivalence, but with the extra assumption that the equations of motion do not depend on the second derivatives of the metric tensor with respect to the space-time coordinates. The above mentioned investigations showed, however, that the particle dynamics is intimately connected with the foundations of the theory of gravitational fields and, at least from a didactical point of view, it may now be of some interest to derive Eq. (1) from the dynamics of a simple model of a clock.

Such a derivation also will allow us to formulate certain requirements as to the construction of a clock in order to make it an *ideal* standard clock in the sense of the general theory of relativity. It will turn out, of course, that a real clock can only approximately be considered ideal and that the degree of accuracy with which it may be said to have this property depends on the properties of the gravitational field in which the clock is placed. For a real clock, the formula (1) is therefore also only approximately true.

In recent years, the construction of accurate time measuring instruments has made great progress and the different "atomic clocks" constructed in various laboratories have an accuracy by far exceeding the accuracy of the earth's rotation. The time does not seem far when the accuracy of such clocks is so high that a direct verification of Eq. (1) is possible by comparison of the rates of two clocks situated at places of different gravitational potential on the earth. It is therefore also of interest to verify that the above mentioned conditions for the validity of the formula (1) are satisfied by these clocks.

These problems are dealt with in Section 6. Section 1 contains a review of the three-dimensional formulation of particle dynamics given elsewhere for gravitational fields with zero vector potentials, while the discussion of the most general case is given in the Appendix. The remaining sections are devoted to mainly didactical remarks illustrating the close relationship between the formulae for the mass and energy of a particle in a gravitational field and the Eq. (1). In Section 2, the formula for the gravitational mass of a particle is illustrated by a discussion of a few "gedanken"-experiments by which this formula, in principle, could be checked. Section 3 presents a similar discussion for the inertial mass. Section 4 brings a derivation of the red-shift formula which is more closely related to the actual mechanism of emission of photons in atomic transitions than usually given in the current text books, the derivation being based directly on a formula describing the influence of gravitational fields on the level scheme of atomic systems. Finally, in Section 5 some of the results obtained in section 3 for the non-relativistic oscillator are derived also for a system with large particle velocities.

1. Particle Dynamics.

The motion of a freely falling particle in an external gravitational field is characterized by the statement that the time track of the particle is a geodesic line. Let us, for simplicity, assume that the system S of space-time coordinates $(x^i) = (x^\iota, ct)$ is time orthogonal*, so that the metric tensor g_{ik}, entering in the interval

$$ds^2 = g_{ik}\, dx^i dx^k, \tag{2}$$

satisfies the three equations

$$g_{\iota 4} = g_{4\iota} = 0. \tag{3}$$

(Latin indices are running from 1 to 4, Greek indices from 1 to 3, only). Then, the spatial line element defining the geometry in the three-dimensional space of our system of reference is simply

$$d\sigma^2 = \gamma_{\iota\varkappa}\, dx^\iota dx^\varkappa, \text{ with } \gamma_{\iota\varkappa} = g_{\iota\varkappa}, \tag{4}$$

and the dynamical action of the gravitational field is determined solely by the scalar gravitational potential $\chi = \chi(x^\iota, t)$ defined by

$$g_{44} = -(1 + 2\chi/c^2). \tag{5}$$

It is now easily seen[5)] that the equations of a geodesic in $(3+1)$-space are equivalent to a set of equations of motion

* By means of the formulae developped in the Appendix all the considerations of the present paper may easily be carried through also in the most general case.

in 3-space, which can be written in the form of a three-dimensional vector equation. Let $\boldsymbol{u}$ be the three-dimensional velocity vector of the particle with the contravariant and covariant components u^{ι} and u_{ι}, respectively, defined by

$$u_{\iota} = g_{\iota\varkappa}\, u^{\varkappa}, \quad u^{\iota} = \frac{dx^{\iota}}{dt}. \tag{6}$$

Further, let $\mathring{m}_0$ denote the proper mass of the particle as measured in a rest system of inertia; then we define the momentum vector $\boldsymbol{p}$ of the particle by the vector equation

$$\boldsymbol{p} = m\,\boldsymbol{u}, \tag{7}$$

where the factor of proportionality—the inertial mass of the particle in the gravitational field—is given by

$$m = \frac{\mathring{m}_0}{\sqrt{1 + 2\,\chi/c^2 - u^2/c^2}} = \mathring{m}_0 \cdot \Gamma. \tag{8}$$

Here, $u^2 = \left(\frac{d\sigma}{dt}\right)^2 = \gamma_{\iota\varkappa}\, u^{\iota} u^{\varkappa}$ is the square of the velocity vector and

$$\Gamma = \frac{1}{\sqrt{1 + 2\,\chi/c^2 - u^2/c^2}} \tag{9}$$

is the generalized Lorentz factor.

The equations of motion then take the form of a three-dimensional vector equation

$$\frac{d_c\, p_{\iota}}{dt} = K_{\iota} \equiv -\, m \frac{\partial \chi}{\partial x^{\iota}}. \tag{10}$$

The left-hand side of this equation is the covariant time derivative of the momentum vector defined by

$$\frac{d_c\, p_{\iota}}{dt} \equiv \frac{dp_{\iota}}{dt} - \frac{1}{2} \frac{\partial \gamma_{\varkappa\lambda}}{\partial x^{\iota}}\, u^{\varkappa} p^{\lambda}, \tag{11}$$

while the right-hand side represents the gravitational force

$$\boldsymbol{K} = -m \operatorname{grad} \chi . \tag{12}$$

The latter is proportional to the negative gradient of the gravitational potential, the factor of proportionality—the gravitational mass—being equal to the inertial mass (8). The last term in (11) is due to the use of curvilinear coordinates in 3-space and is necessary in order to make $\frac{d_c p_\iota}{dt}$ a vector under spatial coordinate transformations.

The equations (10) have the form of usual equations of motion in which the change in the momentum vector $\boldsymbol{p}$ per unit time is equal to the force acting on the particle. They may also be written in Hamiltonian form with the Hamiltonian or the total energy H of the particle in the external gravitational field given by[6)]

$$H = \frac{\overset{\circ}{m}_0 c^2 (1 + 2\chi/c^2)}{\sqrt{1 + 2\chi/c^2 - u^2/c^2}} = m\, c'^2 . \tag{13}$$

Here,

$$c' = c\sqrt{1 + 2\chi/c^2} \tag{14}$$

is the velocity of light $c' = \frac{d\sigma}{dt}$ at a place where the gravitational potential is χ[7)]. Eq. (13) is the generalization of EINSTEIN's energy-mass relation in the presence of gravitational fields. From (7), (8), and (13), we get

$$|\boldsymbol{p}|^2 - \left(\frac{H}{c'}\right)^2 = -\overset{\circ}{m}{}_0^2 c^2 , \tag{15}$$

where $|\boldsymbol{p}|^2 = p_\iota p^\iota$ is the square of the momentum vector. Eq. (15) is the generalization of the usual energy-momentum relation for a free particle. In a static field, where $\chi = \chi(x^\iota)$ is time-independent, the energy H is a constant of the motion.

When the particle is acted upon by a force $\mathfrak{F}$, besides the gravitational force $\boldsymbol{K}$, we have to replace the right-hand side of (10) by the sum $\boldsymbol{K} + \mathfrak{F}$ of the two forces. Hence,[8)]

$$\frac{d_c \boldsymbol{p}}{dt} = \boldsymbol{K} + \mathfrak{F} . \tag{16}$$

For a force $\mathfrak{F}$ of the usual type, which does not change the proper mass $\overset{\circ}{m}_0$, the covariant components $\mathfrak{F}_\iota$ are connected with the generalized Minkowsky four-force F_i by the relation

$$F_i = \left\{ \Gamma F_\iota, -\frac{\Gamma}{c} (F_\varkappa u^\varkappa) \right\}, \tag{17}$$

Γ being the generalized Lorentz factor (9). In a static gravitational field, the energy conservation law takes the form

$$\frac{dH}{dt} = \mathfrak{F} \cdot \boldsymbol{u} = \mathfrak{F}_\iota u^\iota. \tag{18}$$

Derivations of the equations (6)—(18) are found in loc.cit. [5)—8)]. A short derivation of the corresponding relations for the more general case, where (3) does not hold and where therefore the dynamical action of the gravitational field is described by a vector potential as well as by the scalar potential, is found in the Appendix to the present paper.

2. Gravitational Mass.

By putting $u = 0$ in (8) and (13), we get the following expressions for the rest mass m_0 and the rest energy H_0 of a particle in a gravitational field:

$$m_0 = \frac{\overset{\circ}{m}_0}{\sqrt{1 + 2\chi/c^2}}, \tag{19}$$

$$H_0 = \overset{\circ}{m}_0 c^2 \sqrt{1 + 2\chi/c^2} = m_0 \cdot c'^2. \tag{20}$$

Hence, the mass of a body is slightly smaller on the top of a mountain than at sea level, and for the rest energy it is the other way round. Although this variation of the mass is very small (the variation of χ/c^2 is of the order of 10^{-12} between the top of Mount Everest and sea level), it may be of didactical interest to discuss by which experiments the mass (19) in principle could be measured. In this discussion, we shall for simplicity assume that the field is static. Clearly, it will not do to weigh the particle by means of a balance, since the mass of the weights will vary with χ according to exactly the same formula (19). A balance can therefore only be used for measuring the proper mass $\mathring{m}_0$. On the other hand, if the particle is attached to the end of a string of a given length l, the pendulum thus formed will have a period T which will depend solely on the "gravitational acceleration"

$$G = \left| -\operatorname{grad} \chi \right|, \tag{21}$$

at least for small amplitudes and small velocities in the oscillations. Indeed, with these assumptions, the mass m may be treated as a constant, and it will then drop out entirely in the equations of motion (10). Further, if the region in space where the oscillations take place is sufficiently small, we can locally introduce Cartesian coordinates x, y, z (or rather a geodesic system of space coordinates in which the metric tensor $\gamma_{\iota\varkappa}$ may be treated as constant equal to $\delta_{\iota\varkappa}$ inside this region) and, for oscillations along the x-direction, say, the equations of motion (10) reduce to the usual equation of motion for a pendulum

$$\frac{d^2 x}{dt^2} = -\frac{G}{l} x. \tag{22}$$

Hence, we get the usual formula for the period

$$T = 2\pi \sqrt{l/G} \tag{23}$$

and measurements by means of a pendulum can therefore only lead to a determination of the gravitational acceleration or the gradient of χ at an arbitrary point.

In order to measure m_0, it is obviously necessary to use an apparatus in which the particle is acted upon by a non-gravitational force which counterbalances the gravitational force. For instance, we may use a spring-balance, where the non-gravitational force is an elastic force $\mathfrak{F}$ proportional to the elongation s of the spring:

$$\mathfrak{F} = ks. \tag{24}$$

When the spring-balance has come to equilibrium, $\boldsymbol{u} = \dot{\boldsymbol{u}} = 0$, and the left-hand side of (16) vanishes. Thus, we get from (12), (16), (21), and (24) the equation

$$m_0 G = ks, \tag{25}$$

from which we can determine m_0 when G, k, and s are known.

However, it must be noted that the elastic constant k itself depends on the gravitational potential χ according to the formula

$$k = \overset{\circ}{k}\sqrt{1 + 2\,\chi/c^2}, \tag{26}$$

where $\overset{\circ}{k}$ is the value of the elastic constant when the spring is placed at rest in a system of inertia. This fact requires a re-gauging of the spring-balance when it is used at places with different gravitational potentials.

To prove the relation (26) we have simply to make a transformation from the system of coordinates $S: x^i = (x, y, z, ct)$ to a system $\overset{\circ}{S}: (\overset{\circ}{x}_i)$ which is a local system of inertia at rest relative to S at the space-time point considered. The corresponding transformation equations[9] for the Minkowsky four-force are

$$F_\iota = \overset{\circ}{F}_\iota,\ \ F_4 = \overset{\circ}{F}_4\sqrt{1 + 2\,\chi/c^2}. \tag{27}$$

Since $\boldsymbol{u} = \overset{\circ}{\boldsymbol{u}} = 0$ in our case, we get by (17), (9), and (27), remembering that $\overset{\circ}{\chi} = 0$,

$$\mathfrak{F} = \overset{\circ}{\mathfrak{F}}/\Gamma = \overset{\circ}{\mathfrak{F}}\sqrt{1 + 2\,\chi/c^2}, \tag{28}$$

$$ks = \overset{\circ}{k}\overset{\circ}{s}\sqrt{1 + 2\,\chi/c^2}. \tag{29}$$

Further, since the relative velocity of the systems S and $\overset{\circ}{S}$ is zero, we have

$$s = \overset{\circ}{s}, \tag{30}$$

which then leads to the equation (26).

3. Inertial Mass. Harmonic Oscillator.

The mass m_o determined by the equilibrium condition (25) is of course the gravitational mass, the inertial mass entering only in dynamical problems. As an example, we consider small vibrations of the particle attached to the spring-balance around the equilibrium position. For sufficiently small amplitudes and velocities in the vibration, the inertial mass m occurring in the vector $\boldsymbol{p}$ on the left-hand side of the equations of motion (16) may be treated as constant and equal to the m_o given by (19) with the value of χ taken at the equilibrium position. Let us further assume that the elastic constant k is so big that the gravitational force $\boldsymbol{K}$ is negligible compared to the elastic force $\mathfrak{F}$. Finally, we may again, for sufficiently small amplitudes, use local Cartesian coordinates with the x-axis in the direction of the vibration. Then, the equations (16) take the form of the usual equations of motion of a harmonic oscillator

$$m_o \frac{d^2 x}{dt^2} = -kx, \tag{31}$$

x being the distance of the particle from the equilibrium position. A solution of (31) is the harmonic oscillation

$$x = A \sin \omega t \tag{32}$$

with the frequency

$$\omega = \sqrt{k/m_o}. \tag{33}$$

Thus we have, according to (19) and (26),

$$\omega = \sqrt{(\mathring{k}/\mathring{m}_0)(1 + 2\chi/c^2)} = \mathring{\omega}\sqrt{1 + 2\chi/c^2}, \tag{34}$$

where $\mathring{\omega}$ is the frequency of the oscillator when it is placed at rest in a system of inertia. By measuring the frequency of the oscillator when placed at different potentials χ, we get a determination of the inertial mass.

When the oscillating particle carries an electric charge, it emits electromagnetic waves of frequency $\nu = \omega/2\pi$, and (34) then expresses the well-known red-shift of light emitted by a macroscopic oscillating system situated at a place of negative gravitational potential. Of course, the system considered is not a good model of a quantum mechanical system like an atom emitting spectral lines. However, in the following section we shall see that the general formula (20) for the rest energy of a particle in a gravitational field provides a simple derivation of the red-shift formula applicable also to atomic systems.

For small velocities we get for the energy (13) of a particle moving with velocity u in a gravitational field, to the first order in u^2/c^2,

$$H = H_0 + \frac{1}{2} m_0 u^2 \tag{35}$$

with m_0 and H_0 given by (19) and (20). Adding to this the elastic potential energy of the oscillator,

$$V = \frac{1}{2} kx^2, \tag{36}$$

we get the total energy of the oscillator in the gravitational field

$$E = H + V = \mathring{m}_0 c^2 \sqrt{1 + 2\,\chi/c^2} + \varepsilon, \tag{37}$$

where

$$\varepsilon = \frac{1}{2} m_0 u^2 + \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \tag{38}$$

is the usual non-relativistic expression for the energy of an oscillator of mass m_0, elastic constant k and amplitude A.

We shall now compare our oscillator for a given energy state, i. e. a given amplitude A in the gravitational field with the same oscillator placed in a system of inertia. From (37), (38), and (26) we get

$$\varepsilon = \frac{1}{2} \mathring{k} A^2 \sqrt{1 + 2\,\chi/c^2} = \mathring{\varepsilon} \sqrt{1 + 2\,\chi/c^2}, \tag{39}$$

$$E = (\mathring{m}_0 + \mathring{\varepsilon}/c^2)\, c^2 \sqrt{1 + 2\,\chi/c^2} = \mathring{E} \sqrt{1 + 2\,\chi/c^2}, \tag{40}$$

where

$$\mathring{E} = \mathring{m}_0 c^2 + \mathring{\varepsilon}, \quad \mathring{\varepsilon} = \frac{1}{2} \mathring{k} A^2 \tag{41}$$

is the energy of the oscillator in the system of inertia. A comparison of (40) and (20) shows that the oscillator as a whole has the property of a particle at rest in the gravitational field with a proper mass

$$\mathring{M}_0 = \mathring{m}_0 + \mathring{\varepsilon}/c^2, \tag{42}$$

in accordance with EINSTEIN's energy-mass relation. This may serve as an illustration of the fact that the formula (20) for the rest mass of a particle in a gravitational field holds generally, irrespective of the nature of the mass of the particle.

4. The Red-shift of Spectral Lines.

According to BOHR's theory of atomic spectra, the frequency of the light emitted in a transition between two stationary states of the radiating atom is proportional to the difference in energy of the initial and final states. Therefore, from a didactical point of view, it seems most natural to derive the redshift formula by a consideration of the influence of the gravitational potentials on the energy levels of atomic systems. Let $\mathring{E}_1, \mathring{E}_2, \ldots \mathring{E}_n, \ldots$ be the sequence of values of the total energy of the atom in the different stationary states when it is placed at rest in a system of inertia. According to EINSTEIN's mass-energy relation, the proper mass of the atom as a whole in the n'th stationary state is

$$\mathring{M}_{0n} = \mathring{E}_n/c^2. \tag{43}$$

Therefore, by (20), the corresponding energy of the atom when it is placed at rest in a gravitational field must be

$$E_n = \mathring{M}_{0n} c^2 \sqrt{1 + 2\chi/c^2} = \mathring{E}_n \sqrt{1 + 2\chi/c^2}. \tag{44}$$

For the energy release in a transition between two stationary states we then also have

$$\Delta E = \Delta \mathring{E} \sqrt{1 + 2\chi/c^2} \tag{45}$$

which, combined with BOHR's energy-frequency relation

$$\Delta E = h\nu, \quad \Delta \mathring{E} = h\mathring{\nu}, \tag{46}$$

immediately leads to the redshift formula

$$\nu = \mathring{\nu} \sqrt{1 + 2\chi/c^2}. \tag{47}$$

Thus, the energy and frequency of the photons emitted in a definite transition by atoms at the surface of the sun and by terrestrial atoms, for instance, differ by the factor $\sqrt{1+2\chi/c^2}$. On the other hand, a photon traveling from the sun to the earth may be treated as a "freely falling" particle of proper mass zero and velocity $c' = c\sqrt{1+2\chi/c^2}$. Its energy in the static gravitational field is then constantly equal to $h\nu$, which shows that the frequency is unchanged during its travel. After arrival of the photon at the earth, its frequency may be directly compared with the corresponding spectral line emitted by a terrestrial atom.

For $\mathring{m}_0 \rightarrow 0$, we get from (15), for the momentum of the photon,

$$p = \frac{H}{c'} = \frac{h\nu}{c'} = \frac{h}{\lambda}, \tag{48}$$

where $\lambda = c'/\nu$ is the wavelength measured with standard measuring sticks. Further, we get from (13), for the "relativistic" mass m of the photon,

$$m = \frac{h\nu}{c'^2}. \tag{49}$$

If we introduce this value for m into the equations of motion (10), we get, after dividing by the common constant factor $h\nu$,

$$\frac{d_c(u_\iota/c'^2)}{dt} \equiv \frac{d}{dt}(u_\iota/c'^2) - \frac{1}{2}\frac{\partial\gamma_{\varkappa\lambda}}{\partial x^\iota}\frac{u^\varkappa u^\lambda}{c'^2} = -\frac{1}{c'^2}\frac{\partial\chi}{\partial x^\iota} \tag{50}$$

with

$$u^\iota = \frac{dx^\iota}{dt}, \quad u_\iota u^\iota = c'^2. \tag{51}$$

The equations (50) are the equations of motion of a light ray, as derived, for instance, by Fermat's principle[10)], determining the deflection of light in a gravitational field. In this way, the three Einstein effects—the advance of the perihelion, the red-shift of spectral lines, and the deflection of light—appear as consequences of the same equations, the equations of motion (10) which, in turn, may be regarded as a kind of integrability conditions for the gravitational field equations.

5. The "Relativistic" Oscillator.

In the derivation of the relation (34) from the equations of motion, we assumed for convenience that

$$u^2/c'^2 \ll 1. \tag{52}$$

This assumption is of course not essential. As an example let us, as in Section 3, consider the case of a macroscopic particle elastically bound to a fixed point 0; but now we shall not assume that the velocities are small. However, we shall stick to the other assumptions made in Section 3, viz. that the gravitational force can be neglected, and that the potential χ can be regarded as a constant over the small domain of the orbit of the particle. For the mass in the equations of motion we then have, according to (8), (14), and (19)

$$m = \frac{m_0}{\sqrt{1 - u^2/c'^2}}, \tag{53}$$

where m_0 and c' are treated as constants. Using again local Cartesian coordinates, we get now, instead of (31),

$$\frac{d}{dt}\left(\frac{m_0 \dot{x}}{\sqrt{1 - \dot{x}^2/c'^2}}\right) = -kx. \tag{54}$$

The total energy E is a constant of the motion. Hence, by (13), (53), and (36),

$$mc'^2 + V = \frac{m_0 c'^2}{\sqrt{1 - \dot{x}^2/c'^2}} + \frac{1}{2}kx^2 = E. \tag{55}$$

If $\pm A$ are the values of x at the turning points of the particle where $\dot{x} = 0$, the constant E in (55) may be written

$$E = m_0 c'^2 + \frac{1}{2}kA^2, \tag{56}$$

and we get for the velocity $\dot{x}$

$$\left(\frac{dx}{dt}\right)^2 \Big/ c'^2 = 1 - [1 + (A^2 - x^2)(k/2m_0 c'^2)]^{-2}. \tag{57}$$

However, the ratio $k/2\,m_0 c'^2$ is independent of the potential χ. In fact, by (26), (19), and (14),

$$k/2\,m_0 c'^2 = \mathring{k}/2\,\mathring{m}_0 c^2. \tag{58}$$

Therefore, by integration of (57) over one period $T = 1/\nu$, corresponding to a motion of the particle from $x = -A$ to $x = +A$ and back, we get

$$T = \mathring{T}(1 + 2\,\chi/c^2)^{-\frac{1}{2}}, \tag{59}$$

where

$$\mathring{T} = (2/c)\int_{-A}^{A}\left\{1 - [1 + (A^2 - x^2)(\mathring{k}/2\,\mathring{m}_0 c^2)]^{-2}\right\}^{-\frac{1}{2}} dx \tag{60}$$

is the period of the same oscillator when placed at rest in a system of inertia. Eq. (59) or

$$\nu = \mathring{\nu}\sqrt{1 + 2\,\chi/c^2} \tag{61}$$

is identical with (34), which thus has been derived also for a "relativistic" system.

When

$$\mathring{k}A^2/\mathring{m}_0 c^2 \ll 1, \tag{62}$$

Eq. (60) gives of course the non-relativistic expression $\mathring{\nu} = 1/\mathring{T} = \sqrt{\mathring{k}/\mathring{m}_0}/2\,\pi$; which is independent of the amplitude, but in general the frequency will depend on A and be smaller than this value. This is connected with the fact that the relativistic mass m is larger than the rest mass m_0 which will slow down the motion. The velocity u is therefore always smaller than the value $\sqrt{k/m_0}\,A$ for the maximum velocity in the harmonic oscillation (32):

$$u \leqslant \sqrt{k/m_0}\,A. \tag{63}$$

6. Ideal Standard Clocks.

We shall now turn to the problem, mentioned in the Introduction, of deriving the formula (1) for the rate of a clock in a gravitational field from the dynamics of the clock. Let us first consider a clock *at rest* in a *static* gravitational field, in which case we should have, according to (1),

$$d\tau = dt\sqrt{1 + 2\,\chi/c^2}. \tag{64}$$

As a simple model of a clock, we may take the oscillator treated in Section 3, consisting of a particle of proper mass $\mathring{m}_0$, which is elastically bound to a fixed point O in the system of reference, the more so as any vibrating system in a certain approximation has the properties of an oscillator. The time shown by the clock is now per definitionem proportional to the number of beats in the oscillation. The ratio $\omega/\mathring{\omega}$, determined by the equation (34), is therefore equal to the ratio of the rates of the clock when placed at potential χ and at zero potential, respectively. Since the coordinate time t may be identified with the time shown by the clock in the latter case, we see that (64) is a consequence of (34).

We shall now establish the general conditions which a clock must satisfy in order that the formula (1) is valid, i. e. the conditions for the oscillating system to be an ideal standard clock. In the derivation of (34) from the equations of motion (16), we have made a number of assumptions. First, we assumed that the velocity of the particle in the oscillation is small compared with the velocity of light, i. e. that (62) is satisfied. However, as shown in Section 5, this is not a necessary but only a convenient assumption. Next, we made the essential assumption that *the gravitational force $m_0 G$ in (16) could be neglected.* Since the elastic force $\mathfrak{F}$ is of the order kA, the condition for this to be justified is that

$$m_0 G/kA \ll 1. \tag{I}$$

If this condition is not satisfied, the equation of motion for an oscillator in the gravitational field will not have the same form as in a system of inertia. It is true that a constant force $m_0 G$ added to the elastic force of an oscillator will not change the frequency of the oscillator, but only the equilibrium position. However, this holds only for an exact harmonic oscillator; for any potential other than the one given by (36), an additional constant force will change the frequency and invalidate the simple formula (34). To see this, and to get an estimate of this effect, we let $V(x)$ be the potential of the oscillating particle without the force $m_0 G$. The total potential for the system with the additional force is then

$$\overline{V}(x) = V(x) - m_0 G x. \tag{65}$$

If $\overline{x}_0$ and x_0 are the values of x corresponding to the equilibrium positions of the particle with and without the force $m_0 G$, we have

$$V'(x_0) = 0, \tag{66}$$

$$\overline{V}'(\overline{x}_0) = V'(\overline{x}_0) - m_0 G = 0. \tag{67}$$

From the Taylor expansion of $V(x)$ around the point x_0

$$V(x) = V(x_0) + \frac{1}{2} V''(x_0)(x - x_0)^2 + \frac{1}{3!} V'''(x_0)(x - x_0)^3 + \cdots \tag{68}$$

we see that the system without the force $m_0 G$ may be regarded as a harmonic oscillator with the elastic constant

$$k = V''(x_0), \tag{69}$$

provided that the amplitude A satisfies the condition

$$\frac{V'''(x_0) A}{3 V''(x_0)} = \frac{V'''(x_0) A}{3 k} \ll 1. \tag{70}$$

Hence, by (67), (68), and (69),

$$\overline{x}_0 - x_0 = m_0 G/k, \tag{71}$$

which is small compared with A if (I) is satisfied. The system including the constant force $m_0 G$ may therefore be treated as a harmonic oscillator with the elastic constant

$$\left.\begin{aligned} \overline{k} = \overline{V}''(\overline{x}_0) = V''(\overline{x}_0) &= V''(x_0) + V'''(x_0)(\overline{x}_0 - x_0) \\ &= k + V'''(x_0)\, m_0 G/k. \end{aligned}\right\} \tag{72}$$

Here we have used (65), (68) and (71).

The relative change in k due to the gravitational force is thus

$$\delta k/k = V'''(x_0)\, m_0 G/k^2, \tag{73}$$

and the corresponding change in the frequency $\omega = \sqrt{k/m_0}$ is

$$\delta \omega/\omega = \delta k/2k = V'''(x_0)\, m_0 G/2\,k^2. \tag{74}$$

This we may also write

$$\delta \omega/\omega = \frac{3}{2}\left(\frac{V'''(x_0)\, A}{3\,k}\right)\cdot\left(\frac{m_0 G}{kA}\right), \tag{75}$$

showing that the degree of accuracy with which the formula (64) is valid depends not only on how strongly (I) is fulfilled, but also on the degree to which the "harmonicity" condition (70) is satisfied.

In the quantum mechanical derivation of the red-shift formula (47) in Section (4), the assumption (I) would mean that the influence of a constant field of the strength $m_0 G$ on the levels of the atomic system, "the gravitational Stark-effect", is negligible.

In the derivation of (34) from (16) it was further assumed that χ in the expression for the mass on the left-hand side of (16) could be treated as a constant, or more precisely, that

$$\frac{dm_0}{dt}\cdot u \ll \mathfrak{F}. \tag{76}$$

For a static field, this gives, on account of (19), (14), (21), and (63),

$$\frac{\overset{\circ}{m}_0}{(1+2\,\chi/c^2)^{3/2}}\left(\frac{\partial \chi}{\partial x^\iota}u^\iota\right)u/c^2 \approx \frac{m_0\, G u^2}{c'^2} \leqslant \frac{m_0 G\, kA^2}{c'^2 m_0} \ll kA \tag{77 a}$$

or

$$GA/c'^2 \ll 1\,. \tag{II}$$

Since u^2 is always smaller than c'^2, (77 a) or (II) will always hold when condition (I) is satisfied.

For a non-static field, we get from (76) the further condition

$$\frac{m_0\, u}{c'^2}\frac{\partial \chi}{\partial t} \ll kA\,, \tag{77 b}$$

which by (63) and (33) may be written

$$\sqrt{\frac{m_0}{k}}\frac{\partial \chi}{\partial t}\Big/c'^2 = \frac{1}{\omega}\frac{\partial \chi}{\partial t}\Big/c'^2 \ll 1. \tag{III}$$

Thus, the variation of the dimensionless quantity χ/c^2 during one oscillation must be small.

Finally, the use of Cartesian coordinates in the whole region of oscillation, as was implied in the derivation of the oscillator equation (31), is possible only if the spatial curvature can be neglected inside this region. This leads to the conditions

$$\varkappa A^2 \ll 1 \tag{IV}$$

where $\varkappa$ is the Riemann curvature constant of any "plane" surface of geodesics through the point O. For the "plane" defined by the directions of the x^1- and x^2-coordinate curves in O the curvature constant $\varkappa$ is defined by [11)]

$$\varkappa = -P_{1212}/(\gamma_{11}\gamma_{22} - \gamma_{12}^2), \tag{78}$$

where $P_{\iota\varkappa\lambda\mu}$ is the Riemann-Christoffel curvature tensor formed by the spatial metric tensor $\gamma_{\iota\varkappa}$. The corresponding curvatures of the "(23)-" and "(31)-planes" are obtained from (78) by cyclic permutations of the numbers 123.

Up to this point, the centre of the clock, O, has been assumed to be fixed at a definite place in the system of reference. Now, let O be accelerated with the acceleration $\boldsymbol{a}$. As long as the velocity of O is small compared with c', the derivation of (31) from the equations of motion (16) will still be valid if the further condition

$$m_0 a \ll kA \tag{V}$$

is satisfied. Let y^ι and $v^\iota = \dot{y}^\iota$ be the coordinates and velocity of the centre O. For the coordinates and velocity of the particle, we have then

$$x^\iota = y^\iota + \xi^\iota, \quad u^\iota = v^\iota + w^\iota, \tag{79}$$

where ξ^ι is the small vector leading from the centre O to the position of the particle and $w^\iota = \dot{\xi}^\iota$. ($\xi^\iota$ is only approximately a vector!) When (79) is introduced into the left-hand side of (16), it is seen that the equations of motion under the conditions mentioned above again reduce to an equation of the type (31) with x replaced by ξ.

By (33), the condition (V) implies that

$$\Delta v = a/\omega = a\sqrt{m_0/k} \ll \sqrt{k/m_0} \cdot A = \omega A \approx \dot{\xi}\,, \tag{80}$$

i. e. the velocity acquired by O during one period of the oscillation is small compared with the mean velocity of the oscillation. The condition (V) is thus the condition for an "adiabatic acceleration" of the clock. Since (80) implies $\Delta v \ll c'$, this condition also gives the justification for using the simple "action at a distance" expression $\mathfrak{F} = k\xi$ for the force in the equations of motion.

If (I) and (V) are not sufficiently well satisfied, we obviously have to add the extra force $m_0 \boldsymbol{G} - \frac{d_c}{dt}(m_0 \boldsymbol{v}) \approx m_0(\boldsymbol{G} - \boldsymbol{a})$ on the right-hand side of the equations of motion (31). This will cause a change in the frequency which is given by (75), but with G replaced by $|\boldsymbol{G} - \boldsymbol{a}|$. Thus, if the acceleration of the centre of the clock is equal to the gravitational acceleration, as will be the case if the clock is allowed to fall freely, then the two effects dealt with in (I) and (V) will practically cancel and the equations of motion will have the form (31) even if (I) and (V), separately, are not well satisfied.

Finally, when O is moving with the finite velocity $\boldsymbol{v}$, we get again, under the conditions (I)—(V), an equation of the oscillator type (31) for the motion of the particle, but with m_0 replaced by

$$m = \mathring{m}_0 \Gamma(O), \tag{81}$$

where

$$\Gamma(O) = \left\{1 + 2\,\chi/c^2 - v^2/c^2\right\}^{-\frac{1}{2}} \tag{82}$$

is the generalized Lorentz factor corresponding to the velocity of the centre. By a consideration similar to that which in Section 2 lead to the equation (26), we now get for the elastic constant

$$k = \mathring{k}/\Gamma(O), \tag{83}$$

where $\mathring{k}$ is the value of this constant in a local rest system of inertia for the centre O.

Hence, under the above conditions, the frequency ω of an oscillator moving with the velocity $\boldsymbol{v}$ in a gravitational field must be connected with the proper frequency $\mathring{\omega}$ of the same oscillator at rest in a system of inertia by the formula

$$\omega = \mathring{\omega}\sqrt{1 + 2\,\chi/c^2 - v^2/c^2}, \tag{84}$$

in accordance with the formula (1) for an ideal standard clock.

For given G, $\frac{\partial \chi}{\partial t}$, $\varkappa$ and a, it is obviously always possible to choose the parameters of the clock $\mathring{\omega} = \sqrt{\mathring{k}/\mathring{m}_0}$ and A such that the conditions (I) — (V) are satisfied, i. e. it is always possible to construct clocks which are "ideal" under given circumstances. On the other hand, the degree of accuracy to which a given clock (given $\mathring{k}$, $\mathring{m}_0$ and A) may be regarded as ideal depends of course on the use we want to make of it (i. e. on G, $\frac{\partial \chi}{\partial t}$, $\varkappa$ and a). Let us now see to what degree of accuracy the relations (I) — (V) are satisfied by the "atomic clocks" in order to decide whether the variations in the rate of the clock due to variations in the gravitational field of the earth could in principle be measured by means of such clocks. It is a common feature of these clocks that atomic systems like ammonia molecules act as the "balance" of the clock. The vibrations of the atoms in the molecule, which in this connection may be treated as a classical mechanical system, are to a high degree of accuracy harmonic oscillations. The frequency of the oscillation is of the order

$$\omega \approx 10^{10}\,sec^{-1}. \tag{85}$$

Since the mass of the oscillating particle is of the order

$$m_0 \approx 10^{-24}\,gr \tag{86}$$

the system may be represented by a harmonic oscillator with an elastic constant

$$k = \omega^2 m_0 \approx 10^{-4}\,gr/\text{sec}^2. \tag{87}$$

The amplitude A of the oscillator cannot be larger than atomic dimensions. In the conditions I, and II we can therefore put

$$A \approx 10^{-8}\,\text{cm}, \quad c' \approx c = 3\cdot 10^{10}\,\text{cm/sec}, \quad G \approx 10^{3}\,\text{cm/sec}^2 \qquad (88)$$

for terrestrial gravitational fields.

Hence,

$$m_0\, G/k\, A \approx 10^{-9} \qquad (89)$$

$$GA/c'^2 \approx 10^{-25}. \qquad (90)$$

Further, a straight-forward calculation shows that the quantity on the left hand side of (IV), for a point at the surface of the earth, is of the order $\frac{\partial^2 \chi}{\partial r^2}\frac{A^2}{c^2} \approx (GA/c'^2)\cdot(A/r)$ where r is the radius of the earth.

The conditions (I) — (V) are therefore amply satisfied and the condition (V) is of course also well satisfied even for accelerations considerably larger than the gravitational acceleration. We also see that the oscillator is highly non-relativistic, since

$$\omega^2 A^2/c'^2 \approx 10^{-17} \ll 1. \qquad (91)$$

From (75) and (89) we now get

$$\delta\omega/\omega \approx 10^{-9}\,(V'''(x_0)\,A/3\,k). \qquad (91)$$

Thus, if we are aiming at an accuracy of the order of 10^{-12}, the quantity on the left-hand side of (70) which determines the degree of harmonicity of the oscillator must be smaller than 10^{-3}. However, as pointed out on p. 22, the accuracy to which the clock may be considered "ideal" increases considerably if the atomic systems which constitute the balance of the clock are freely falling in the gravitational field, since the effects dealt with in (I) and (V) then almost completely cancel.

In concluding, I wish to thank Dr. D. Frisch for pleasant and illuminating discussions on problems connected with the atomic clocks.

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Appendix.

In this Appendix, we shall give a short derivation of the three-dimensional equations of motion in the most general case, where (3) is not satisfied and where the dynamical action of the gravitational field is described by a vector potential

$$\gamma_\iota = g_{\iota 4}/\sqrt{-g_{44}} \tag{A. 1}$$

as well as by the scalar potential defined by (5). In this case, we have, instead of (4), the spatial metric tensor $\gamma_{\iota\varkappa}$ given by

$$d\sigma^2 = \gamma_{\iota\varkappa}\, dx^\iota\, dx^\varkappa, \quad \gamma_{\iota\varkappa} = g_{\iota\varkappa} + \gamma_\iota \gamma_\varkappa. \tag{A. 2}$$

Let

$$d\tau = ds/ic = \sqrt{-g_{ik}\, dx^i\, dx^k}/c \tag{A. 3}$$

be the real quantity measuring the length of the time track of a particle in (3 + 1)-space, and let F_i be the covariant components of the non-gravitational four-force. Then, using an arbitrary parameter representation, the equations of the time track may be derived from the variational principle

$$\delta \int_{\lambda_1}^{\lambda_2} m_0 c \sqrt{-g_{ik}\, \dot{x}^i\, \dot{x}^k}\, d\lambda - \int_{\lambda_1}^{\lambda_2} F_i\, \delta x^i \frac{d\tau}{d\lambda}\, d\lambda = 0, \tag{A. 4}$$

where δx^i are arbitrary variations of the space-time coordinates x^i vanishing for $\lambda = \begin{cases} \lambda^2 \\ \lambda^1 \end{cases}$, and

$$\dot{x}^i = \frac{dx^i}{d\lambda} = \frac{dx^i}{d\tau} \cdot \frac{d\tau}{d\lambda}. \tag{A. 5}$$

Since λ may be chosen arbitrarily, we can, for instance, use the time coordinate $t = x^4/c$ as parameter, in which case $\delta x^4 = 0$ in (A. 4) and $\dot{x}^\iota = \dfrac{dx^\iota}{dt} = u^\iota$ is the three-dimensional velocity. Further, in this case one easily gets

$$\frac{d\tau}{d\lambda} = \frac{1}{c}\sqrt{-g_{ik}\, \dot{x}^i\, \dot{x}^k} = \left[\left(\sqrt{1 + 2\chi/c^2} - \gamma_\iota u^\iota/c\right)^2 - u^2/c^2\right]^{\frac{1}{2}} = 1/\Gamma, \tag{A. 6}$$

where Γ is the Lorentz factor in this most general case, and $u^2 = \gamma_{\iota\varkappa} u^\iota u^\varkappa$. With this expression for Γ, the connection between F_i and the components $\mathfrak{F}_\iota$ of the three-dimensional force is then again given by (17).

With this choice of λ, the variational principle (A. 4) takes the form

$$\int_{t_1}^{t_2} \left\{ \delta L(x^\iota, u^\iota) + \mathfrak{F}_\iota \, \delta x^\iota \right\} dt = O, \tag{A. 7}$$

where

$$L(x^\iota, u^\iota, t) = - \mathring{m}_0 c \sqrt{(c' - \gamma_\varkappa u^\varkappa)^2 - u^2} \tag{A. 8}$$

is the Langrangian of the particle in the gravitational field. Here, $x^\iota = x^\iota(t)$ as a function of the time t is varied in such a way that $\delta x^\iota = O$ for $t = \begin{cases} t_1 \\ t_2 \end{cases}$. In the expression for the Lagrangian, $c' = c\sqrt{1 + 2\chi(x^\iota, t)/c^2}$ is the quantity introduced by (14), but in the present case, $(\gamma_\iota \neq O)$, the velocity of light depends on the direction of propagation and c' is now the velocity of light in a direction perpendicular to the space vector γ_ι[7].

As in (8), the mass of the particle in the gravitational field is defined as $m = \mathring{m}_0 \Gamma$, but with the Lorentz factor given by (A. 6), i. e.

$$m = \mathring{m}_0 \Gamma = \frac{\mathring{m}_0}{\left[\left(\sqrt{1 + 2\chi/c^2} - \gamma_\varkappa u^\varkappa/c\right)^2 - u^2/c^2\right]^{1/2}} \tag{A. 9}$$

is now a function of the four potentials (γ_ι, χ) as well as of the velocity $\boldsymbol{u}$. For the canonically conjugate momentum to the coordinate x^ι, we thus get, by (A. 8) and (A. 9),

$$\pi_\iota = \frac{\partial L}{\partial \dot{x}^\iota} = m u_\iota + m\gamma_\iota (c' - \gamma_\varkappa u^\varkappa) = p_\iota + m\gamma_\iota (c' - \gamma_\varkappa u^\varkappa). \tag{A. 10}$$

Thus, π_ι differs from the momentum $p_\iota = m u_\iota$ of the particle by a term depending on the vector potential in analogy with the case of a particle in an electromagnetic field.

The equations of motion following from the variational principle (A. 7) are

$$\frac{d\pi_\iota}{dt} = \frac{\partial L}{\partial x^\iota} + \mathfrak{F}_\iota. \tag{A. 11}$$

By (A. 9) the Lagrangian (A. 8) can also be written

$$L = -m\left[(c' - \gamma_\varkappa u^\varkappa)^2 - u^2\right]. \tag{A. 12}$$

Thus, by (A. 10) and (A. 12), we get for the Hamiltonian H corresponding to the Lagrangian L

$$\left.\begin{aligned} H \equiv \pi_\iota \dot{x}^\iota - L = m\,(u_\iota u^\iota) + m\,(\gamma_\iota u^\iota)\,[c' - \gamma_\varkappa u^\varkappa] \\ + m\,[(c' - \gamma_\varkappa u^\varkappa)^2 - u^2], \end{aligned}\right\} \tag{A. 13}$$

which leads to the following expression for the energy of the particle in the gravitational field:

$$H = mc'\,(c' - \gamma_\varkappa u^\varkappa). \tag{A. 14}$$

In the special case $\gamma_\iota = O$, Eqs. (A. 9), (A. 14) are identical with the equations (8), (13) in Section 1.

By using the definitions (A. 10, 13) of π_ι and H, and the equations of motion (A. 11), we get for the time derivative of H

$$\frac{dH}{dt} = \dot{\pi}_\iota u^\iota + \pi_\iota \dot{u}^\iota - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial x^\iota} u^\iota - \frac{\partial L}{\partial u^\iota} \dot{u}^\iota = \mathfrak{F}_\iota u^\iota - \frac{\partial L}{\partial t}, \tag{A. 15}$$

and, by (A. 14), π_ι in (A. 10) may be written

$$\pi_\iota = p_\iota + H\gamma_\iota/c'. \tag{A. 16}$$

Thus, the left-hand side of (A. 11) takes the form

$$\frac{d\pi_\iota}{dt} = \frac{dp_\iota}{dt} + \left(\mathfrak{F}_\varkappa u^\varkappa - \frac{\partial L}{\partial t}\right)\gamma_\iota/c' + H\frac{d}{dt}(\gamma_\iota/c').$$

Further, we get by a simple calculation from (A. 8) and (A. 14)

$$\frac{\partial L}{\partial x^\iota} = \frac{1}{2}\frac{\partial \gamma_{\varkappa\lambda}}{\partial x^\iota} u^\varkappa u^\lambda m - \frac{H}{c'}\left(\frac{\partial c'}{\partial x^\iota} - \frac{\partial \gamma_\varkappa}{\partial x^\iota} u^\varkappa\right) \tag{A. 17}$$

so that the equation of motion may be written

$$\left.\begin{aligned} \frac{d_G p_\iota}{dt} = \mathfrak{F}_\iota - (\mathfrak{F}_\varkappa u^\varkappa)\,\gamma_\iota/c' + \frac{\partial L}{\partial t}\gamma_\iota/c' - H\frac{d}{dt}(\gamma_\iota/c') \\ - \frac{H}{c'}\left(\frac{\partial c'}{\partial x^\iota} - \frac{\partial \gamma_\varkappa}{\partial x^\iota} u^\varkappa\right). \end{aligned}\right\} \tag{A. 18}$$

By this explicit use of the conservation law (A. 15) for the energy, we have achieved that the right-hand side of (A. 18), the force acting on the particle, does not contain the acceleration, but only the coordinates and the velocity of the particle. This expression for the force can be simplified by introducing the antisymmetric spatial tensor $\omega_{\iota\varkappa}$ which is connected to the local rotation of our system of reference with respect to the local systems of inertia. The latter is defined by[12)]

$$\omega_{\iota\varkappa}/c = \left(\frac{\partial}{\partial x^\iota} + \frac{\gamma_\iota}{c'}\frac{\partial}{\partial t}\right)(\gamma_\varkappa/c') - \left(\frac{\partial}{\partial x^\varkappa} + \frac{\gamma_\varkappa}{c'}\frac{\partial}{\partial t}\right)(\gamma_\iota/c'). \qquad \text{(A. 19)}$$

By a somewhat lengthy, but elementary calculation, (A. 18) can be written in the form

$$\frac{d_c p_\iota}{dt} = \mathfrak{F}_\iota - (\mathfrak{F}_\varkappa u^\varkappa)\,\gamma_\iota/c' + mG_\iota \qquad \text{(A. 20)}$$

with

$$\left.\begin{aligned} G_\iota &= -(1-\gamma_\varkappa u^\varkappa/c')^2\left(\frac{\partial\chi}{\partial x^\iota} + c'\frac{\partial\gamma_\iota}{\partial t}\right) \\ &+ \sqrt{1+2\,\chi/c^2}\cdot(c'-\gamma_\varkappa u^\varkappa)\,\omega_{\iota\lambda}u^\lambda + \frac{1}{2}\frac{\gamma_\iota}{c'}\frac{\partial\gamma_{\varkappa\lambda}}{\partial t}u^\varkappa u^\lambda. \end{aligned}\right\} \qquad \text{(A. 21)}$$

If we put $\gamma_\iota = O$, these equations are reduced to the simple equation (16). Further, in the case of *stationary weak fields*, where time derivations and products of the potentials γ_ι and χ can be neglected, we get for the gravitational force

$$K_\iota = m\left(-\frac{\partial\chi}{\partial x^\iota} + c\,\omega_{\iota\varkappa}u^\varkappa\right), \qquad \omega_{\iota\varkappa} = \frac{\partial\gamma_\varkappa}{\partial x^\iota} - \frac{\partial\gamma_\iota}{\partial x^\varkappa}. \qquad \text{(A. 22)}$$

In a rigid system rotating with constant angular velocity relative to a system of inertia, the equation (A. 22) for K_ι gives the usual expressions for the centrifugal and Coriolis forces which therefore are valid for arbitrary velocities of the particle[13)], the only effect of relativity in the equations of motion being the velocity dependence of the mass according to the formula (A. 9).

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5) C. M. Chapt. X, § 110, p. 290 and C. M. Appendix **7,** p. 378; further the Appendix of the present paper.
6) C. M. Appendix **7,** p. 378.
7) See f. inst. C. M. Chapt. VIII, § 89, Eq. (70).
8) C. M. Chapt. X, § 112.
9) See f. inst. C. M. Chapt. IX, § 104.
10) Eq. (50) is identical with Eq. (98) in C. M. Chapt. X, § 117, if the arbitrary parameter in the latter equation is chosen equal to the time variable.
11) See f. inst. W. Pauli, Relativitätstheorie, Enzyklopädie d. mathem. Wissensch. **V 19** § 17.
12) C. M. Chapt. VIII, § 94, Eq. (110).
13) In C. M. Chapt. X, § 110, p. 292, this was proved only for small velocities.

Indleveret til selskabet den 15. juni 1955.
Færdig fra trykkeriet den 3. oktober 1955.